

# Set Systems and Families of Permutations with Small Traces\*

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## Abstract

Let  $F$  be a family of permutations on  $[n] = \{1, \dots, n\}$  and let  $Y = \{y_1, \dots, y_m\} \subseteq [n]$ , with  $y_1 < y_2 < \dots < y_m$ . The *restriction* of a permutation  $\sigma$  on  $[n]$  to  $Y$  is the permutation  $\sigma|_Y$  on  $[m]$  such that  $\sigma|_Y(i) < \sigma|_Y(j)$  if and only if  $\sigma(y_i) < \sigma(y_j)$ ; the restriction of  $F$  to  $Y$  is  $F|_Y = \{\sigma|_Y \mid \sigma \in F\}$ . Marcus and Tardos proved the well-known conjecture of Stanley and Wilf that for any permutation  $\tau$  on  $[m]$  there is a constant  $c$  such that if no permutation in  $F$  admits  $\tau$  as a restriction then  $F$  has size  $O(c^n)$ . In the same vein, Raz proved that there is a constant  $C$  such that if the restriction of  $F$  to any *triple* has size at most 5 (regardless of what these restrictions are) then  $F$  has size at most  $C^n$ . In this paper, we consider the following natural extension of Raz's question: assuming that the restriction of  $F$  to any  $m$ -element subset in  $[n]$  has size at most  $k$ , how large can  $F$  be?

We first investigate a similar question for set systems. A *set system* on  $X$  is a collection of subsets of  $X$  and the *trace* of a set system  $R$  on a subset  $Y \subseteq X$  is the collection  $R|_Y = \{e \cap Y \mid e \in R\}$ . For finite  $X$ , we show that if for any subset  $Y \subset X$  of size  $b$  the size of  $R|_Y$  is smaller than  $2^i(b - i + 1)$  for some integer  $i$  then  $R$  consists of  $O(|X|^i)$  sets. This generalizes Sauer's Lemma on the size of set systems with bounded VC-dimension. We show that in certain situations, bounding the size of  $R$  knowing the size of its restriction on all subsets of small size is equivalent to *Dirac-type problems* in extremal graph theory. In particular, this yields bounds with non-integer exponents on the size of set systems satisfying certain trace conditions.

We then map a family  $F$  of permutations on  $[n]$  to a set system  $R$  on the pairs of  $[n]$  by associating each permutation to its set of inversions. Conditions on the number of restrictions of  $F$  thus become conditions on the size of traces of  $R$ . Our generalization of Sauer's Lemma and bounds on certain Dirac-type problems then yield a delineation, in the  $(m, k)$ -domain, of the main growth rates of  $F$  as a function of  $n$ .

## 1 Introduction

In this paper, we study two problems of the following flavor: how large can a family of combinatorial objects defined on a finite set be if its number of distinct “projections” on any small subset is bounded? We consider set systems, where the “projection” is the standard notion of trace, and families of permutations, where the “projection” corresponds to the notion of containment used in the study of permutations with excluded patterns. One of our motivations for considering these questions is the “geometric permutation problem” in *geometric transversal theory*, a question that has been open for two decades, and which we explain in the conclusion.

**Set systems.** A *set system*, also called a *range space* or a *hypergraph*, is a pair  $(X, R)$  where  $X$  is a set, the *ground set*, and  $R$  is a set of subsets of  $X$ , the *ranges*. Given  $Y \subseteq X$ , the *trace* of  $R$  on  $Y$ , denoted  $R|_Y$ , is the set  $\{e \cap Y \mid e \in R\}$ . Given an integer  $b$ , let  $\binom{X}{b}$  denote the set of  $b$ -element subsets of  $X$ , and define:

$$f_R(b) = \max_{Y \in \binom{X}{b}} |R|_Y|.$$

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The function  $f_R$  is called the *shatter function* of  $(X, R)$ , and counts the size of the largest trace on a subset of  $X$  of size  $b$ . Let  $\text{Sh}(n, b, k)$  denote the maximum number of ranges in a set system  $(X, R)$  on  $|X| = n$  elements with  $f_R(b) \leq k$ . The first problem we consider is the following:

**Question 1.** Given  $b$  and  $k$ , how large is  $\text{Sh}(n, b, k)$ ?

For  $k = 2^b - 1$ , the answer is given by Sauer's Lemma [28] (also proven independently by Perles and Shelah [30] and Vapnik and Chervonenkis [32]), which states that

$$\text{Sh}(n, b, 2^b - 1) = \sum_{i=0}^{b-1} \binom{n}{i} = O(n^{b-1}). \quad (1)$$

The largest  $b$  such that  $f_R(b) = 2^b$  is known as the *VC-dimension* of  $([n], R)$ . The theory of set systems of bounded VC-dimension, and in particular Sauer's Lemma, has many applications, in particular in geometry and approximation algorithms; classical examples include the  $\varepsilon$ -net Theorem [10] or improved approximation algorithms for geometric set cover [8].

For the case of graphs, that is, set systems where all ranges have size 2, Question 1 is a classical problem known as a *Dirac-type problem*: determining the maximum number  $\text{Ex}(n, m, \mu)$  of edges in a graph on  $n$  vertices whose induced subgraph on any  $m$  vertices has at most  $\mu$  edges. These problems were extensively studied in extremal graph theory since the 1960's, and we refer to the survey of Griggs, Simonovits and Thomas [18] for an overview. For the case of general set systems, Chapter 11 of Jukna's book [20] gives an overview. In particular, Frankl [14] proved that

$$\text{Sh}(n, 3, 6) = t_2(n) + n + 1 \quad \text{and} \quad \text{Sh}(n, 4, 10) = t_3(n) + n + 1,$$

where  $t_i(n)$  denotes the number of edges of the Turán<sup>1</sup> graph  $T_i(n)$ . Bollobás and Radcliffe [6] showed that

$$\text{Sh}(n, 4, 11) \leq \binom{n}{2} + n + 1 \quad \text{except for } n = 6.$$

There has also been interest in the case where  $b = \alpha n$  and  $b = n - \Theta(1)$ ; we refer to Jukna's book or the article of Bollobás and Radcliffe for these results.

**Permutations.** Let  $\sigma$  be a permutation on  $[n] = \{1, \dots, n\}$  and let  $Y = \{y_1, \dots, y_m\} \subseteq [n]$ , with  $y_1 < y_2 < \dots < y_m$ . The *restriction* of  $\sigma$  to  $Y$  is the permutation  $\sigma|_Y$  on  $[m]$  such that  $\sigma|_Y(i) < \sigma|_Y(j)$  if and only if  $\sigma(y_i) < \sigma(y_j)$ . This allows to define the *shatter function* of a set  $F$  of permutations:

$$\phi_F(m) = \max_{Y \in \binom{[n]}{m}} |F|_Y|,$$

where  $F|_Y = \{\sigma|_Y \mid \sigma \in F\}$  denotes the set of restrictions of permutations of  $F$  to  $Y$ . Raz [26] proposed to extend the notion of VC-dimension to sets of permutations by defining the VC-dimension of  $F$  as the largest  $m$  such that  $\phi_F(m) = m!$ , and the analogue of Question 1 arises naturally for sets of permutations:

**Question 2.** Given  $m$  and  $k$ , how large can a set  $F$  of permutations on  $[n]$  be if  $\phi_F(m) \leq k$ ?

Raz [26] showed that any family of permutations on  $[n]$  such that  $\phi_F(3) < 6$  has size at most exponential in  $n$ , and asked whether the same holds whenever  $k < m!$ . Cibulka and Kynčl [12] answered this question negatively by showing that the maximum size of a family  $F$  of permutations on  $[n]$  with  $\phi_F(4) < 4!$  is  $2^{\Theta(n \log \alpha(n))}$ , where  $\alpha$  denotes the inverse of the Ackermann function; they also give upper and lower bounds on the maximum size of a family of permutations with VC-dimension  $m$  for all  $m$ , which are tight for every even  $m$ .

Question 2 is related to classical problems on families of permutations with an *excluded pattern*. We say that a permutation  $\sigma$  on  $[n]$  *contains* a permutation  $\tau$  on  $[m]$  if there is an  $m$ -element subset  $Y \subseteq [n]$

<sup>1</sup>The Turán graph  $T_i(n)$  has  $n$  vertices, partitioned into  $i$  subsets, each of size  $\lfloor n/i \rfloor$  or  $\lceil n/i \rceil$ , with an edge between any two vertices from different subsets.

such that  $\sigma|_Y = \tau$ . If no permutation in a family  $F$  of permutations on  $[n]$  contains  $\tau$ , then  $F$  *avoids*  $\tau$  and  $\tau$  is an *excluded pattern* for  $F$ . The study of families of permutations with excluded patterns goes back to a work of Knuth [23], motivated by sorting permutations using stacks, and received considerable attention over the last decades. In particular, Stanley and Wilf asked whether for any fixed permutation  $\tau$  the number of permutations on  $[n]$  that avoid  $\tau$  is at most exponential in  $n$ , a question answered in the positive by Marcus and Tardos [25]. If a family of permutations avoids a permutation  $\tau$  on  $[m]$ , then its restriction to any  $m$ -element subset  $Y \subseteq [n]$  has size less than  $m!$ , and so its VC-dimension is at most  $m - 1$ . In this sense Raz's question generalizes that of Stanley and Wilf.

**Our results.** We present three results. First, we generalize Sauer's Lemma, and show that if  $0 \leq i < b \leq n$  then  $\text{Sh}(n, b, 2^i(b - i + 1) - 1) \leq (b - i) \sum_{0 \leq j \leq i} \binom{n}{j} = O(n^i)$  (Theorem 1). Then, we prove that the condition  $f_R(b) = k$  is, when asymptotic orders of magnitude are considered, *equivalent* to a Dirac-type problem on graphs for  $b < k < 4(b - 1)$  (Lemma 2); this implies that  $\text{Sh}(n, b, k) = \Theta(n^\alpha)$  with non-integer  $\alpha$  for certain values of  $k$ , a behavior not captured by Theorem 1. Finally, we give a reduction of the permutation problem to the set system problem (Lemma 4) from which we deduce the main transitions between the constant, polynomial and at least exponential behaviors for Question 2; we also give a simple condition for propagating exponential upper bounds (Lemma 6).

## 2 Sauer's Lemma for set systems with small traces

Our first result is the following generalization of Sauer's Lemma (which is the special case  $i = b - 1$ ):

**Theorem 1.** *Let  $(X, R)$  be a set system. For any  $0 \leq i < b \leq n$ , if  $f_R(b) < 2^i(b - i + 1)$  then  $f_R(n) \leq (b - i) \sum_{0 \leq j \leq i} \binom{n}{j}$ .*

*Proof.* Define  $v_i(b) = 2^i(b - i + 1)$  and consider the property:

$$P(b, i, n) : \quad f_R(b) < v_i(b) \Rightarrow f_R(n) \leq (b - i) \sum_{0 \leq j \leq i} \binom{n}{j}.$$

We prove by induction that  $P(b, i, n)$  holds for all choices of integers  $0 \leq i < b \leq n$ .

*Initialization.* Observe that for any  $0 \leq i < b$ , we have

$$(b - i) \sum_{0 \leq j \leq i} \binom{b}{j} \geq (b - i) \sum_{0 \leq j \leq i} \binom{i + 1}{j} = (b - i)(2^{i+1} - 1) \geq 2^i(b - i) + 2^i - 1 = v_i(b) - 1,$$

so  $P(b, i, b)$  holds for any  $i < b$ . Next, we show that  $P(b, 0, n)$  holds for all  $b, n \geq 1$ . We first argue, by induction, that for  $b \geq 1$ , for any  $b + 1$  pairwise distinct subsets  $S_1, \dots, S_{b+1}$  of  $X$  there exists  $Y \subset X$  of size at most  $b$  such that the restrictions  $S_j \cap Y$  are already pairwise distinct.<sup>2</sup> This is obvious for  $b = 1$ . Assume it is true for  $b - 1$ , so that there exists a subset  $Y_0$  of size  $b - 1$  for which  $S_1 \cap Y_0, \dots, S_b \cap Y_0$  are pairwise distinct; if  $S_{b+1} \cap Y_0$  is distinct from all other restrictions then we are done, else  $S_{b+1} \cap Y_0 = S_j \cap Y_0$  for exactly one index  $j$ ; then, for any element  $y \in (S_j \setminus S_{b+1}) \cup (S_{b+1} \setminus S_j)$  the sets  $S_1, \dots, S_{b+1}$  have pairwise distinct restrictions on  $Y_0 \cup \{y\}$ , which is of size at most  $b$ . As a consequence we have that for all  $b, n$ ,

$$f_R(b) < b + 1 = v_0(b) \Rightarrow f_R(n) \leq b = (b - 0) \binom{n}{0},$$

and  $P(b, 0, n)$  holds for all  $b, n \geq 1$ .

*Induction.* We can now adapt the induction of Sauer's original proof to show that if, given  $i < b \leq n$ , property  $P(b', i', n')$  holds for all triples  $(b', i', n')$  lexicographically smaller than  $(b, i, n)$ , then  $P(b, i, n)$  also holds. So let  $(X, R)$  be a set system such that  $f_R(b) < v_i(b)$  and let  $Y$  be an  $n$ -element subset of  $X$ . We pick an element  $a \in Y$  and let  $D$  denote the ranges in  $R_{|Y \setminus \{a\}}$  that are the trace of two distinct ranges from  $R|_Y$ . Since  $f_R(b) < v_i(b)$ , it is clear that  $f_{R|_{Y \setminus \{a\}}}(b) < v_i(b)$ . Moreover, if  $|D| \geq v_{i-1}(b - 1)$

<sup>2</sup>A fact sometimes referred to as Bondy's Theorem [5].

$b \backslash  R $	$\Theta(1)$	$\Theta(n)$	$\Theta(n^2)$	$\Theta(n^3)$	$\Theta(n^4)$	$\Theta(n^5)$
2	1-2	3				
3	1-3	4-5	6-7			
4	1-4	5-7	9-11	12-15		
5	1-5	6-9	12-15	18-23	24-31	
6	1-6	7-11	16-19	27-31	36-47	48-63
7	1-7	8-13	20-23	36-39	54-63	$\emptyset$

Table 1: Values of  $k$  for which the condition  $\lambda_i(b) \leq k < 2^i(b-i+1)$  indicates that the order of magnitude of the bound of Theorem 1 on  $\text{Sh}(n, b, k)$  is tight.

for some subset  $Z \in \binom{Y}{b-1}$  then  $|R_{|Z \cup \{y\}}| \geq 2v_{i-1}(b-1) = v_i(b)$  so  $f_R(b) < v_i(b)$  also implies that  $f_D(b-1) < v_{i-1}(b-1)$ . Since

$$|R_Y| = |R_{Y \setminus \{a\}}| + |D|,$$

the induction hypothesis ensures that

$$|R_{Y \setminus \{a\}}| \leq (b-i) \sum_{0 \leq j \leq i} \binom{n-1}{j} \quad \text{and} \quad |D| \leq (b-i) \sum_{0 \leq j \leq i-1} \binom{n-1}{j},$$

we have:

$$|R_Y| \leq (b-i) + (b-i) \sum_{1 \leq j \leq i} \binom{n-1}{j} + (b-i) \sum_{1 \leq j \leq i} \binom{n-1}{j-1}.$$

Then, from  $\binom{n-1}{j} + \binom{n-1}{j-1} = \binom{n}{j}$  we get that for any  $n$ -element subset  $Y$  of  $X$ ,

$$|R_Y| \leq (b-i) + (b-i) \sum_{1 \leq j \leq i} \binom{n}{j} = (b-i) \sum_{0 \leq j \leq i} \binom{n}{j},$$

and  $P(b, i, n)$  holds. This concludes the proof.  $\square$

For an integer  $x$ , let  $P_i(x)$  denote the set of all partitions of  $x$  into a sum of  $i$  non-negative integers and define:

$$\lambda_i(b) = \max_{\{b_1, \dots, b_i\} \in P_i(b)} \prod_{j=1}^i (b_j + 1).$$

Consider the following set system  $(X, R)$ : split the finite ground set  $X$  into  $i$  roughly equal subsets  $X_1, \dots, X_i$  and let  $R$  denote the set of all  $i$ -element subsets containing exactly one element from each  $X_j$ . We have  $f_R(b) \leq \lambda_i(b)$  and  $f_R(n) = \Theta(n^i)$ . This implies that for any  $k$  such that  $\lambda_i(b) \leq k < 2^i(b-i+1)$  we have  $\text{Sh}(n, b, k) = \Theta(n^i)$ , and so the order of magnitude given by Theorem 1 is tight. Table 1 summarizes these cases for small values of  $b$ . In particular, Theorem 1 gives the correct order of magnitude of  $\text{Sh}(n, b, k)$  for all  $b \leq 4$ , with the exception of  $\text{Sh}(n, 4, 8)$ . We will see in the next section that  $\text{Sh}(n, 4, 8) = \Theta(n\sqrt{n})$ .

**Remark.** Theorem 1 asserts, in particular, that for any set system  $(X, R)$  with  $f_R(b) < 2^i(b-i+1)$  we have  $f_R(n) = O(n^i)$ . The condition  $f_R(b) < 2^i(b-i+1)$  does not, however, imply that  $R$  has VC-dimension at most  $i$ . A simple example is given by

$$X = [n] \quad \text{and} \quad R = \{A \mid A \subset [i]\} \cup \{\{x\} \mid x \in [n]\},$$

which has VC-dimension  $i$  and for which  $f_R(b) = 2^i + b - i$  is smaller than  $2^{i-1}(b-i)$  for  $b$  large enough.

### 3 Fractional exponents from Dirac-type bounds

Recall that  $\text{Ex}(n, m, \mu)$  denotes the maximum number of edges in a graph on  $n$  vertices whose induced subgraph on any  $m$  vertices has at most  $\mu$  edges.

**Lemma 2.** *If  $0 \leq b < k \leq n$  then  $\text{Sh}(n, b, k) \geq n + 1 + \text{Ex}(n, b, k - b - 1)$  and if  $k < 4(b - 1)$  then  $\text{Sh}(n, b, k) \leq n + 1 + f_R(b) + k \cdot \text{Ex}(n, b, k - b - 1)$ .*

*Proof.* Let  $G = (V, E)$  be a graph on  $n$  vertices such that any  $b$  vertices span at most  $\mu$  edges and such that  $|E| = \text{Ex}(n, b, \mu)$ . Observe that some  $b$  vertices of  $E$  must span exactly  $\mu$  edges by maximality of  $\text{Ex}(n, b, \mu)$ . Putting  $(X, R) = (V, \{\emptyset\} \cup V \cup E)$  we obtain a hypergraph such that  $f_R(b) = \mu + b + 1$ , proving the lower bound:

$$\text{Sh}(n, b, k) \geq n + 1 + \text{Ex}(n, b, k - b - 1).$$

The proof of the upper bound uses two claims: (i) the upper bound in the general case follows from the special case where  $R$  is closed under taking subsets, and (ii) if  $R$  is closed under taking subsets, the condition  $f_R(b) < 4(b - 1)$  implies that the number of subsets in  $R$  of size three or larger is at most  $f_R(b) - 1$  times the number of subsets in  $R$  of size two.

Claim (i) is a direct consequence of the classical lemma<sup>3</sup> of Alon [1] and Frankl [14] stating that for any set system  $(X, R)$  there exists a set system  $(X, \tilde{R})$  such that  $\tilde{R}$  is closed under taking subsets,  $|\tilde{R}| = |R|$ , and  $f_{\tilde{R}}(b) \leq f_R(b)$  for every  $b$ .

We now prove claim (ii). Assume that  $R$  is closed under taking subsets and let  $e \in R$  be a 2-element subset. Assume that there are  $d$  subsets in  $R$  that contain  $e$ , and let  $F$  denote the union of these subsets. We claim that  $|F| < b$ ; assume, by contradiction, that  $|F| \geq b$  and let  $G$  be a  $b$ -element subset of  $F$  containing  $e$ . Writing  $e = \{p_1, p_2\}$  we get that  $R$  contains the sets  $\emptyset, \{p_1\}, \{p_2\}, \{p_1, p_2\}$  as it is closed under taking subsets. Furthermore, every  $p \in G \setminus e$  is part of a subset that contains  $e$ , and therefore contains  $e \cup \{p\}$ . It follows that  $R$  also contains the four ranges  $\{p\}, \{p, p_1\}, \{p, p_2\}$ , and  $\{p, p_1, p_2\}$  for every  $p \in G \setminus e$ . Thus, if  $|F| \geq b$  then  $|R|_G \geq 4 + 4(b - 2) > f_R(b)$ , a contradiction; it must then be that  $|F| < b$  and thus:

$$d \leq |R|_F \leq f_R(|F|) \leq f_R(b - 1) \leq f_R(b) - 1.$$

In other words, every pair  $e \in R$  is contained in at most  $f_R(b) - 1$  elements of  $R$ , and claim (ii) follows.

Now, let  $(X, R)$  be a set system such that  $|X| = n$  and  $f_R(b) = k$  with  $k < 4(b - 1)$ . By claim (i) we can assume that  $R$  is closed under taking subsets. After discarding all elements of  $X$  that do not appear in a subset in  $R$  we have  $|X| \leq n$  and still  $f_R(b) = k$ . For  $t \geq 0$  let  $R_t$  denote the family of subsets in  $R$  of size  $t$ . It is clear that  $|R_0| = 1$  and  $|R_1| = |X| \leq n$ . It follows from claim (ii) that  $|\bigcup_{t \geq 3} R_t| \leq (k - 1)|R_2|$ . Now, if  $|X| < b$  then  $|R| \leq f_R(b) = k$  and the upper bound holds. Otherwise,  $(X, R_2)$  is a graph on at most  $n$  vertices where any  $b$  vertices span at most  $k - b - 1$  edges, so  $|R_2| \leq \text{Ex}(n, b, k - b - 1)$  and the upper bound follows.  $\square$

Let us recall a few bounds on  $\text{Ex}(n, m, \mu)$  that we use here and in Section 4; once again, we refer to the survey of Griggs, Simonovits and Thomas [18] for a more detailed overview. (Recall that when  $H$  is a graph,  $\text{Ex}(n, H)$  denotes the maximum number of edges in a graph on  $n$  vertices that does not have  $H$  as a subgraph.)

- $\text{Ex}(n, m, m - 2)$  is  $O(n)$ , as a graph for which any  $m$  vertices span at most  $m - 2$  edges has only connected components of size at most  $m - 1$ .
- $\text{Ex}(n, m, m - 1)$  is  $O(n^{1+1/\lfloor m/2 \rfloor})$  [7] and is superlinear if  $m \geq 3$  [24].
- If  $m \geq s + t$  and  $k < st$  then  $\text{Ex}(n, m, k) \leq \text{Ex}(n, K_{s,t}) = O(n^{2-1/s})$  by the Kővári-Sós-Turán Theorem [22] (sharper bounds were obtained by Füredi [15]).

Also, note that  $\text{Ex}(n, 2rp - 2p + 2, 2rp - p) < 6prn^{1+1/r}$  (a consequence of [9, Lemma 1]). When  $\mu$  is close to  $m$ , this implies a stronger bound on  $\text{Ex}(n, m, \mu)$  than the three results above, for instance for  $r = 3$  and  $p = 2$  we get  $\text{Ex}(n, 10, 10) = O(n^{4/3})$ . These implications hold for values of  $m$  that are somewhat large, so we do not elaborate on them. Let  $z_s(n)$  denote the maximum number of edges in a  $K_{s,s}$ -free bipartite graph on  $n$  vertices. It is known that  $z_2(n) = \Theta(n\sqrt{n})$  and that  $z_3(n) = \Theta(n^{5/3})$  (see

<sup>3</sup>See Jukna [20, Chapter 11], Bollobás [5, Chapter 17] and the survey of Füredi and Pach [16] for more details.

Blagojević, Bukh and Karasev [4] for a discussion of these and related results). For small values of  $n$ ,  $z_2(n)$  was studied by Goddard et al. [17] and  $z_3(n)$  was studied by Guy [19].

**Corollary 3.** *For  $b \geq 4$ , we have:*

- (i) *If  $s \geq 2$  and  $k \geq z_s(b) + b + 1$ , then  $\text{Sh}(n, b, k) \geq n + 1 + z_s(n)$ .*
- (ii) *If  $s \geq 2$ ,  $k < 4(b - 1)$ , and  $k < (s + 1)(b - s + 1)$ , then  $\text{Sh}(n, b, k) = O(n^{2-1/s})$ .*
- (iii)  *$\text{Sh}(n, b, 2b)$  is  $O(n^{1+2/b})$  and is superlinear.*

*Proof.* (i) Given a bipartite graph  $G = (V, E)$  that is  $K_{s,s}$ -free for  $s \geq 2$ , the set system  $(X, R) = (V, \{\emptyset\} \cup V \cup E)$  satisfies  $f_R(b) \leq b + 1 + z_s(b)$ . Since  $|R| = 1 + n + z_s(n)$ , claim (i) follows.

(ii)  $k < 4(b - 1)$  implies by Lemma 2 that  $\text{Sh}(n, b, k) = \Theta(n + \text{Ex}(n, b, k - b - 1))$ . Since

$$k - b - 1 < (s + 1)(b - s + 1) - b - 1 = s(b - s)$$

we have  $\text{Ex}(n, b, k - b - 1) \leq \text{Ex}(n, K_{s,b-s}) = O(n^{2-1/s})$ .

(iii) By Lemma 2 we have  $\text{Sh}(n, b, 2b) = \Theta(n + \text{Ex}(n, b, b - 1))$  and claim (iii) follows from the known bounds on  $\text{Ex}(n, b, b - 1)$ .  $\square$

**Remark.** The cases  $s = 2$  and  $s = 3$  of Corollary 3 are particularly interesting as they lead to the following tight bounds for  $b \geq 4$ :

$$\text{Sh}(n, b, k) = \Theta(n\sqrt{n}) \quad \text{for } z_2(b) + b + 1 \leq k \leq 3b - 4,$$

$$\text{Sh}(n, b, k) = \Theta(n^{5/3}) \quad \text{for } z_3(b) + b + 1 \leq k \leq 4b - 9.$$

From the values  $z_2(4) = 3$ ,  $z_2(5) = 4$  and  $z_2(6) = 6$  [17, Table 3] and  $z_3(6) = 8$  and  $z_3(7) = 10$  [19], we deduce, in particular, that

$$\text{Sh}(n, b, k) = \Theta(n\sqrt{n}) \quad \text{for } (b, k) \in \{(4, 8), (5, 10), (5, 11), (6, 13), (6, 14)\}, \quad \text{and}$$

$$\text{Sh}(n, b, k) = \Theta(n^{5/3}) \quad \text{for } (b, k) \in \{(6, 15), (7, 18), (7, 19)\}.$$

## 4 Families of permutations

In this section we give bounds on the size of a family  $F$  of permutations on  $[n]$  with a given  $\phi_F(b)$ .

**Reduction to set systems.** An *inversion* of a permutation  $\sigma$  on  $[n]$  is a pair  $1 \leq i < j \leq n$  such that  $\sigma(i) > \sigma(j)$ . The *distinguishing pair* of two permutations  $\sigma_1$  and  $\sigma_2$  is the lexicographically smallest pair  $(i, j)$  that is an inversion for one but not the other. If  $F$  is a family of permutations on  $[n]$  we let  $I_F$  denote the set of distinguishing pairs for all pairs of permutations from  $F$ . Given a permutation  $\sigma \in F$ , we let  $R(\sigma)$  denote the set of elements of  $I_F$  that are inversions of  $\sigma$ , and let  $R(F) = \{R(\sigma) \mid \sigma \in F\}$ ; see Fig. 1 for an example. We observe that  $(I_F, R(F))$  is a range space and that  $R$  is a one-to-one map between  $F$  and  $R(F)$ . In particular  $|F| = |R(F)|$ .

**Lemma 4.** *If  $F$  is a family of permutations on  $[n]$  and  $m \geq 2$  then  $f_{R(F)}(\lfloor \frac{m}{2} \rfloor) \leq \phi_F(m)$  and  $|I_F| \leq \text{Ex}(n, m, \phi_F(m) - 1)$ .*

*Proof.* Let  $b = \lfloor \frac{m}{2} \rfloor$ , put  $k = f_{R(F)}(b)$ , let  $Y \subseteq I_F$  be a set of  $b$  distinguishing pairs such that  $|R(F)_{|Y}| = k$ , and let  $\sigma_1, \dots, \sigma_k$  be permutations in  $F$  such that  $R(F)_{|Y} = \{R(\sigma_1)_{|Y}, \dots, R(\sigma_k)_{|Y}\}$ . We set  $Z = \bigcup_{p \in Y} p$  as the subset of elements  $[n]$  that appear in a pair in  $Y$ .  $R(\sigma_i)_{|Y} \neq R(\sigma_j)_{|Y}$  implies  $\sigma_i|_Z \neq \sigma_j|_Z$ , and so the permutations  $\sigma_1|_Z, \dots, \sigma_k|_Z$  are pairwise distinct. This implies that  $\phi_F(|Z|) \geq k$  and as  $|Z| \leq 2b \leq m$ , the first statement follows.

Let  $s(t)$  denote the maximum number of distinguishing pairs in a family of  $t$  permutations on  $[n]$ . We have  $s(2) = 1$ , and the following inequality follows from the distinguishing tree structure illustrated in Fig. 1:

$$s(t) \leq 1 + \max_{1 \leq i \leq t-1} \{s(i) + s(t-i)\},$$

$\sigma_1 = 1234567; \quad R(\sigma_1) = \emptyset$   
 $\sigma_2 = 1235764; \quad R(\sigma_2) = \{(4, 5)\}$   
 $\sigma_3 = 2134567; \quad R(\sigma_3) = \{(1, 2)\}$   
 $\sigma_4 = 2143765; \quad R(\sigma_4) = \{(1, 2), (3, 4)\}$   
 $\sigma_5 = 2347561; \quad R(\sigma_5) = \{(1, 2), (1, 3)\}$   
 $\sigma_6 = 2354761; \quad R(\sigma_6) = \{(1, 2), (1, 3), (4, 5)\}$

$$I_F = \{(1, 2), (1, 3), (3, 4), (4, 5)\}$$

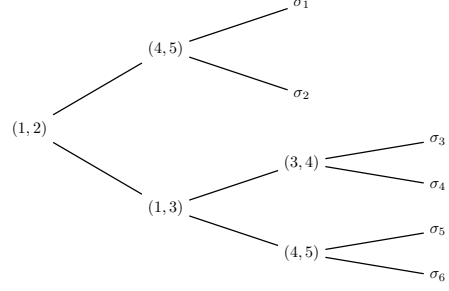


Figure 1:  $I_F$  and the elements of  $R(F)$  for a set  $F$  of six permutations on the set  $[7]$ . On the right, its distinguishing tree: its root is the lexicographically smallest distinguishing pair  $(i, j)$ , the upper subtree is recursively built with the permutations that do not invert  $(i, j)$  and the lower subtree is recursively built with the permutations that invert  $(i, j)$ .

By induction we obtain  $s(t) \leq t - 1$ . This implies that in the graph  $G = ([n], I_F)$ , any  $m$  vertices span at most  $\phi_F(m) - 1$  edges, and it follows that

$$|I_F| \leq \text{Ex}(n, m, \phi_F(m) - 1),$$

which concludes the proof. □

**Main transitions.** We can now outline the main transitions in the growth rate of families of permutations according to the value of  $\phi_F(m)$ . Let  $b = \lfloor \frac{m}{2} \rfloor$ .

**Constant.** If  $\phi_F(m) \leq \lfloor \frac{m}{2} \rfloor$  then, by Lemma 4,  $f_{R(F)}(b) \leq b$  and Theorem 1 with  $i = 0$  yields that  $|F| = |R(F)| = O(1)$ .

**Linear.** Assume that  $\lfloor \frac{m}{2} \rfloor < \phi_F(m) < 2\lfloor \frac{m}{2} \rfloor$ . Then, by Lemma 4,  $f_{R(F)}(b) < 2b$  and Theorem 1 with  $i = 1$  yields that  $|F| = |R(F)| = O(|I_F|) = O(\text{Ex}(n, m, m - 2)) = O(n)$ . A matching lower bound is given by the family

$F_1$  : the identity and all permutations on  $[n]$  that differ from the identity by the transposition of a single pair of the form  $(2i, 2i + 1)$ .

$F_1$  has size  $1 + \lfloor \frac{n}{2} \rfloor$  and  $\phi_{F_1}(m) = \lfloor \frac{m}{2} \rfloor + 1$ .

**Polynomial.** If  $\phi_F(m) < 2\lfloor \frac{m}{2} \rfloor$  then, by Lemma 4,  $f_{R(F)}(b) < 2^b$  and  $(I_F, R(F))$  has VC-dimension at most  $b - 1$ . It follows, from Sauer's Lemma, that  $|F| = |R(F)| = O(|I_F|^{b-1})$ , and since  $|I_F| = O(n^2)$ , we get that  $|F|$  is  $O(n^{2\lfloor \frac{m}{2} \rfloor - 2})$ . This bound can of course be refined using the results of Sections 2 and 3 (see Table 3).

**At least exponential.** If  $\phi_F(m) \geq 2\lfloor \frac{m}{2} \rfloor$  then  $|F|$  can be exponential in  $n$ . An example is the family

$F_2$  : all permutations on  $[n]$  that differ from the identity by the transposition of any number of pairs of the form  $(2i, 2i + 1)$ .

$F_2$  has size  $2^{\lfloor \frac{n}{2} \rfloor}$  and  $\phi_{F_2}(m) = 2^{\lfloor \frac{m}{2} \rfloor}$ .

In particular, any shattering condition either forces the size of the family of permutations to be polynomial or allows it to be exponential. A similar dichotomy was observed for growth rates of families of permutations with excluded patterns: the set of permutations on  $[n]$  avoiding a given family of patterns grows either at most polynomially or at least exponentially in  $n$  [21]. A similar result, albeit with a weaker bound, follows easily from our tabulation:

**Corollary 5.** Let  $\Sigma$  be a set of permutations (possibly of different sizes) and for  $n \in \mathbb{N}$  let  $\text{Forb}_n(\Sigma)$  denote the set of size- $n$  permutations that contain no element of  $\Sigma$  as a restriction. Either there is a constant  $c$ , depending on  $\Sigma$ , such that  $|\text{Forb}_n(\Sigma)| \leq n^c$  for all  $n \geq 1$  or  $|\text{Forb}_n(\Sigma)| \geq 2^{\lfloor \frac{n}{2} \rfloor}$  for all  $n \geq 1$ .

*Proof.* Assume that we are not in the latter case and there exists  $n_0$  such that  $|\text{Forb}_{n_0}(\Sigma)| < 2^{\lfloor \frac{n_0}{2} \rfloor}$ . By definition of  $\text{Forb}_n(\Sigma)$ , for any  $1 \leq \alpha \leq \beta$  and any  $\alpha$ -element subset  $S \subseteq [\beta]$ , we have  $\text{Forb}_\beta(\Sigma)|_S \subseteq \text{Forb}_\alpha(\Sigma)$ . In particular, for any  $n \geq n_0$ , any restriction of  $\text{Forb}_n(\Sigma)$  to an  $n_0$ -element subset has size at most  $|\text{Forb}_{n_0}(\Sigma)| < 2^{\lfloor \frac{n_0}{2} \rfloor}$ . It follows that  $\phi_{\text{Forb}_n(\Sigma)}(n_0) < 2^{\lfloor \frac{n_0}{2} \rfloor}$  and the size of  $\text{Forb}_n(\Sigma)$  is  $O\left(n^{2^{\lfloor \frac{n_0}{2} \rfloor - 2}}\right)$ .  $\square$

Although  $\text{Ex}(n, m, m-1)$  is superlinear, we have not found an example where  $|I_F|$  or  $F$  have super-linear size when  $\phi_F(m) = m$ . The main transitions are summarized in Table 2.

	$\phi_F(m) \leq \lfloor \frac{m}{2} \rfloor$	$\lfloor \frac{m}{2} \rfloor < \phi_F(m) < 2\lfloor \frac{m}{2} \rfloor$	$2\lfloor \frac{m}{2} \rfloor \leq \phi_F(m) < 2^{\lfloor \frac{m}{2} \rfloor}$	$2^{\lfloor \frac{m}{2} \rfloor} \leq \phi_F(m)$
$ F $	$\Theta(1)$	$\Theta(n)$	$\Omega(n)$ and $O\left(n^{2^{\lfloor \frac{m}{2} \rfloor - 2}}\right)$	$\Omega(2^{\lfloor \frac{n}{2} \rfloor})$

Table 2: Maximum size of a family  $F$  of permutations as a function of  $\phi_F(m)$ .

**Exponential upper bounds.** Raz [26] proved that if  $\phi_F(3) \leq 5$  then  $|F|$  has size at most exponential in  $n$ . The following simple observation derives similar bounds for a few other values of  $\phi_F(m)$ .

**Lemma 6.** *If, for given integers  $m$  and  $k$  there is  $C > 0$  such that  $\phi_F(m-1) \leq k-1$  implies  $|F| \leq C^n$  for a family  $F$  of permutations on  $[n]$ , then  $\phi_F(m) \leq k$  implies  $|F| \leq D^n$  for  $D = \max\{C, k\}$ .*

*Proof.* We prove the statement by induction over  $n$ . It is trivial for  $n = m$ , as then  $|F| = \phi_F(n) = \phi_F(m) \leq k$ . So assume  $n > m$  and let  $F$  be a family of maximum size with  $\phi_F(m) \leq k$ . If  $\phi_F(m-1) < k$  then the assumption implies  $|F| \leq C^n \leq D^n$ . We can thus assume  $\phi_F(m-1) = k$ .

Let  $X \in \binom{[n]}{m-1}$  such that  $F|_X = \{\sigma_1, \dots, \sigma_k\}$  has size  $k$ , and let

$$F^i = \{\sigma \in F \mid \sigma|_X = \sigma_i\}.$$

$F$  is the disjoint union of the  $F^i$ . For any  $e \in [n] \setminus X$  and any  $i = 1, \dots, k$ , the family  $F^i$  restricts to a single permutation on  $X \cup \{e\}$  since  $\phi_F(m) = k$ . Let  $a \in X$ , and let  $Y = [n] \setminus \{a\}$ . By the above, for two distinct permutations  $\sigma, \tau \in F^i$  we have  $\sigma|_Y \neq \tau|_Y$ . This implies that the family  $G^i = F^i|_Y$  has  $|G^i| = |F^i|$ . But  $G^i$  is a family of permutations on  $[n-1]$  with  $\phi_{G^i}(m) \leq k$ , and so by the induction assumption we have  $|G^i| \leq D^{n-1}$ . It follows that

$$|F| \leq \sum_{i=1}^k |F^i| = \sum_{i=1}^k |G^i| \leq k \cdot D^{n-1} \leq D^n,$$

since  $k \leq D$ .  $\square$

With Raz's result that  $\phi_F(3) \leq 5$  implies that  $|F|$  is exponentially bounded, we obtain that  $|F|$  is exponentially bounded whenever  $\phi_F(m) \leq m+2$ . Table 3 tabulates our results for small values of  $m$  and  $\phi_F(m)$ .

## 5 Conclusion

A natural open question is the tightening of the bounds for both Questions 1 and 2. In particular, the first case where Lemma 4 no longer guarantees that the reduction from permutations to set systems leads to a ground set with linear size is  $\phi_F(m) = m$ ; does that condition still imply that  $|I_F|$  or  $|F|$  are  $O(n)$  when  $m$  is large enough?

How much does Raz's condition need to be weakened, that is, for which values of  $m$  and  $k$  does  $\phi_F(m) \leq k$  imply that  $|F|$  is at most exponential in  $n$ ? Can this question be tackled by a shifting technique similar to the lemma of Alon and Frankl used in the proof of Lemma 2?

A line intersecting a collection  $C$  of pairwise disjoint convex sets in  $\mathbb{R}^d$  induces two permutations, one reverse of the other, corresponding to the order in which each orientation of the line meets the sets.



$m \backslash k$	3	4	5	6	7	8	9	10
3	$2^{\Theta(n)}$	$2^{\Theta(n)}$	$2^{\Theta(n)}$	$n!$	-	-	-	-
4	$\lfloor \frac{2n}{3} \rfloor + 1$	$2^{\Theta(n)}$	$2^{\Theta(n)}$	$2^{\Theta(n)}$	$2^{\Omega(n)}$	$2^{\Omega(n)}$	$2^{\Omega(n)}$	$2^{\Omega(n)}$
5	$\lfloor \frac{n}{2} \rfloor + 1$	$2^{\Theta(n)}$	$2^{\Theta(n)}$	$2^{\Theta(n)}$	$2^{\Theta(n)}$	$2^{\Omega(n)}$	$2^{\Omega(n)}$	$2^{\Omega(n)}$
6	3	$\Theta(n)$	$\Theta(n)$	$O(n^{8/3})$	$O(n^3)$	$2^{\Theta(n)}$	$2^{\Omega(n)}$	$2^{\Omega(n)}$
7	3	$\Theta(n)$	$\Theta(n)$	$O(n^2)$	$O(n^{8/3})$	$2^{\Theta(n)}$	$2^{\Theta(n)}$	$2^{\Omega(n)}$
8	3	4	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$O(n^{15/8})$	$O(n^3)$	$O(n^3)$
9	3	4	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$O(n^{3/2})$	$O(n^{5/2})$	$O(n^3)$
10	3	4	5	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$O(n^{9/5})$

$ I_F  \leq \text{Ex}(n, m, m-2) = O(n)$	$ I_F  \leq \text{Ex}(n, m, m-1) = O\left(n^{1+\lfloor \frac{1}{m} \rfloor}\right)$	$ I_F  \leq \text{Ex}(n, m, 2m-5) = O(n\sqrt{n})$
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Table 3: Maximum size of a family  $F$  of permutations on  $[n]$  with  $\phi_F(m) = k$ . We use the bound on  $\text{Sh}(|I_F|, \lfloor \frac{m}{2} \rfloor, k)$  from Sections 2 and 3, and the shading of the cell indicates which of the three bounds on  $\text{Ex}(n, m, \mu)$  mentioned in Section 3 is used to control the size of  $I_F$ . For example, for  $m = 8$  and  $k = 8$  we have  $|I_F| \leq \text{Ex}(n, 8, 7) = O(n^{5/4})$  and  $|F| \leq \text{Sh}(|I_F|, 4, 8) = O(|I_F|^{3/2}) = O(n^{15/8})$ .

The pair of these permutations is called a *geometric permutation* of  $C$ . Let  $g(d, n)$  denote the maximum number of geometric permutations of a collection of  $n$  pairwise disjoint sets in  $\mathbb{R}^d$ . While the exact value of  $g(2, n)$  is known to be  $2n - 2$  [13], determining the asymptotic order of magnitude of  $g(d, n)$  for  $d \geq 3$  has been one of the main open questions in geometric transversal theory [33] for the last two decades (see for instance [2, 3, 11, 29]). The upper bound on  $g(d, n)$  was recently improved to  $O(n^{2d-3} \log n)$  [27], and the best known lower bound is  $\Omega(n^{d-1})$  [31]. We can pick from each geometric permutation one of its elements so that the resulting family  $F$  has the following property: if any  $m$  members of  $C$  have at most  $k$  distinct geometric permutations then  $\phi_F(m-2) \leq k$ . One interesting question is whether bounds such as the one we obtained could lead to new results on the geometric permutation problem.

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